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A Characterization of $\mathscr{P}_1$ Spaces*

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A Banach space X is $\mathscr{P}_1$ if and only if the radius of each bounded set in X is half its diameter.

Let A be a bounded subset of a Banach space. Its radius is defined as

$$r(A) = \inf\{\rho \mid S(x, \rho) \supset A\},$$

where $S(x, \rho)$ denotes the closed sphere of radius ρ about x. Its diameter is

$$\delta(A) = \sup\{\| x - y \| \mid x, y \in A\}.$$

A Banach space X is $\mathscr{P}_1$ if X is norm 1 complemented in every Banach space $Z \supset X$. It is known (see, e.g., [3, p. 193]) that if X is $\mathscr{P}_1$, then $\delta(A) = 2r(A)$ for every bounded $A \subset X$. Here we show that the converse statement is also true, that is

THEOREM. *If $\delta(A) = 2r(A)$ for every bounded subset A of a Banach space X, then X is $\mathscr{P}_1$.*

The theorem extends the known characterizations of $\mathscr{P}_1$ spaces due to Nachbin (see [4]), Grünbaum [2], and Lindenstrauss [4].

To see this, notice that $\delta(A) = 2r(A)$ for every bounded A in X if and only if X has the following intersection property for spheres.

IP3. For every family $\{S(x_\alpha ; 1)\}$ of pairwise intersecting spheres, and for every $\epsilon > 0$, $\cap S(x_\alpha ; 1 + \epsilon) \neq \varnothing$.

Nachbin showed that X is $\mathscr{P}_1$ if and only if it has

IP1. For every family $\{S(x_\alpha, \rho_\alpha)\}$ of pairwise intersecting spheres, $\cap S(x_\alpha, \rho_\alpha) \neq \varnothing$.

Lindenstrauss, following the work of Grünbaum, showed that X is $\mathscr{P}_1$ if and only if X has

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IP2. For every family $\{S(x_\alpha\,;\rho_\alpha)\}$ of pairwise intersecting spheres, and for every $\epsilon > 0$, $\bigcap S(x_\alpha\,;\rho_\alpha(1+\epsilon)) \neq \varnothing$.

In [1, Sect. 3, Theorem 4] the following intersection property (which differs from IP2 in the expansion of the spheres for large values of ρ_α) was shown to be equivalent to IP1.

IP$\frac{1}{2}$. For every family $\{S(x_\alpha\,;\rho_\alpha)\}$ of pairwise intersecting spheres, and for every $\epsilon > 0$, $\bigcap S(x_\alpha\,;\rho_\alpha+\epsilon) \neq \varnothing$.

Grunbaum [2] showed that X has IP2 if and only if, for every $\epsilon > 0$ and every $Y \supset X$ with $\dim(Y/X) = 1$, there is a projection of Y onto X with norm $\leqslant 1+\epsilon$. Lindenstrauss [4, Theorem 6.10] then showed that such an X must be a $\mathscr{P}_1$ space. Our strategy, then, is to show that, if X has IP3, then for every $Y \supset X$ with $\dim(Y/X) = 1$, and for every $\epsilon > 0$, there is a projection of Y onto X with norm $\leqslant 1+\epsilon$. It would be somewhat more pleasant to show directly that, with IP3, there is already a norm 1 projection of Y onto X, since then a standard Zorn's lemma argument would show directly that X is $\mathscr{P}_1$. This would provide a proof of the theorem without appeal to the result of Lindenstrauss, and, in fact, provide an alternative proof of his theorem.

Proof of the theorem. We first show that, if X has IP3, then every collection $\{S(x_\alpha\,;1)\}$ of pairwise intersecting spheres has a nonempty intersection. Let $\epsilon_n \searrow 0$ with $\epsilon_1 < 1$. If $y_1 \in \bigcap S(x_\alpha\,;1+\epsilon_1)$, the triangle inequality shows that $S(y_1\,;\epsilon_1)$ meets each of the spheres $S(x_\alpha\,;1)$. Let

$$\{x_\alpha^{(1)}\} = \{x_\alpha\} \cup Z_1, \qquad \text{where } Z_1 = \{z : \|z - y_1\| \leqslant 1-\epsilon_1\}.$$

Now, $\bigcap\{S(z,1) \mid z \in Z_1\} = S(y_1, \epsilon_1)$, so it follows that $\{S(x_\alpha^{(1)};1)\}$ is a pairwise intersecting family. Proceed inductively in this way to produce families $\{x_\alpha\} \subset \{x_\alpha^{(1)}\} \subset \{x_\alpha^{(2)}\} \subset \cdots$ so that, for each n, $\bigcap S(x_\alpha^{(n-1)}, 1+\epsilon_n) \supset \bigcap S(x_\alpha^{(n)}, 1+\epsilon_{n+1}) \neq \varnothing$. Since

$$\bigcap S(x_\alpha^{(n)}, 1+\epsilon_{n+1}) \subset \bigcap\{S(z; 1+\epsilon_{n+1}) \mid z \in Z_n\} = S(y_n\,;\epsilon_n+\epsilon_{n+1}),$$

the diameter of $\bigcap S(x_\alpha^{(n)}; 1+\epsilon_{n+1})$ goes to zero as $n \to \infty$. It is immediate that $\bigcap_n \bigcap_\alpha S(x_\alpha^{(n)}; 1+\epsilon_{n+1}) \subset S(x_\alpha\,;1)$, which is, therefore, nonempty. Thus if $A = \{x_\alpha\}$ has diameter 2, there is a sphere of radius 1 containing A. We are now in a position to construct the desired projections. Let Y be an arbitrary Banach space with $Y \supset X$ and $\dim(Y/X) = 1$. Let $f \in Y^*$, $\|f\| = 1$, $f(X) = 0$. Let $X_\alpha = f^{-1}(\alpha)$, which for each α is a hyperplane in Y, isometric to X. With $S(y,\rho)$ denoting the sphere of radius ρ about y in Y, let $B_\alpha = S(0;1) \cap X_\alpha$ for $0 \leqslant \alpha < 1$, so that each B_α is a set in X_α with $\delta(B_\alpha) \leqslant 2$. By the first part of the proof, $C_\alpha = \bigcap\{S(x, 1) \mid z \in B_\alpha\} \cap X_\alpha \neq \varnothing$.

We now show that for any $z_\alpha \in C_\alpha$, the projection P_α of Y onto X along

z_α (defined by $P_\alpha(u) = u - (f(u)/\alpha) z_\alpha$) has the property that $\| P_\alpha u \| \leqslant 1$ if $\| u \| \leqslant 1$ and $f(u) \geqslant \alpha$. First, if $f(u) = \alpha$, and $\| u \| \leqslant 1$, we have

$$P_\alpha(u) = u - z_\alpha ,$$

which has norm $\leqslant 1$ by the selection of z_α. Now, with α fixed, define $\varphi_\beta : X_\alpha \to X_\beta$ by

$$\varphi_\beta(y) = y + \left(\frac{\beta - \alpha}{\alpha}\right) z_\alpha$$

for each $\beta > \alpha$. Notice that the point y is on the segment joining $P_\alpha(y)$ and $\varphi_\beta(y)$ for each y in X_α. Thus, if $y \in B_\alpha$, $\| y \| = 1$, since $\| P_\alpha(y) \| \leqslant 1$, we have $\| \varphi_\beta(y) \| \geqslant 1$. Since $\varphi_\beta(X_\alpha) = X_\beta$, it follows by convexity that $\varphi_\beta(B_\alpha) \supset B_\beta$: Consider the cylinder

$$\mathscr{C} = \bigcup\{[P_\alpha z, \varphi_1(z)] \mid z \in B_\alpha\}.$$

$\mathscr{C}$ is a convex body with boundary

$$\{P_\alpha(z) \mid z \in B_\alpha\} \cup \{\varphi_1(z) \mid z \in B_\alpha\} \cup \bigcup\{[P_\alpha(z), \varphi_1(z)] \mid z \in B_\alpha , \| z \| = 1\},$$

and any $y \in B_\alpha$ with $\| y \| < 1$ is in the interior of $\mathscr{C}$. Further, $\varphi_\beta(B_\alpha) = \mathscr{C} \cap X_\beta$. Now suppose that $x \in X_\beta \backslash \varphi_\beta(B_\alpha)$ with $\| x \| < 1$. For any $y \in B_\alpha$ with $\| y \| < 1$, the segment $[y, x]$ contains only vectors of norm <1, but it must cross the lateral boundary $(\bigcup\{[P_\alpha(z), \varphi_\beta(z)] \mid z \in B_\alpha , \| z \| = 1\})$ of $\mathscr{C}$ at a point u with $f(u) > \alpha$. This forces $\| u \| \geqslant 1$, which is a contradiction. Since $\varphi_\beta(B_\alpha) \supset B_\beta$,

$$P_\alpha(B_\beta) \subset P_\alpha(\varphi_\beta(B_\alpha)) = P_\alpha(B_\alpha) \subset S_X(0; 1).$$

Thus, $\| u \| \leqslant 1, f(u) \geqslant \alpha$ forces $\| P_\alpha(u) \| \leqslant 1$. To complete the proof, simply observe that

$$P_\alpha(u) = u - \frac{f(u)}{\alpha} z_\alpha = u - \frac{f(u)}{\beta} \varphi_\beta(z_\alpha),$$

so that by choosing β close to 1 and noting that $\| \varphi_\beta(z_\alpha) \| \leqslant 2$, $\| P_\alpha(u) \| \leqslant 1 + (2/\beta) \mid f(u) \mid$ if $\| u \| \leqslant 1$. Thus, for $0 \leqslant f(u) \leqslant \alpha$, and $\alpha < \beta\epsilon/2$, we have $\| P_\alpha(u) \| \leqslant 1 + \epsilon$.

This, combined with the remarks preceding the present proof, establishes our theorem.

Remark. It should be noted that one can devise a somewhat simpler proof of the theorem modeled after Grünbaum's proof [2], which stays in the context of intersection properties. However, the author prefers this version which seems to display more of the geometric difficulty encountered in trying to avoid the use of Lindenstrauss' theorem.

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